Shapley-Scarf Markets with Objective Indifferences

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Abstract

In many object allocation problems, some of the objects may be indistinguishable from each other. For example, in a college dormitory, rooms in the same building with the same floor plan are effectively identical. In such cases, it is reasonable to assume that agents are indifferent between identical objects, and matching mechanisms in these settings should account for the agents' indifferences. Top trading cycles (TTC) with fixed tie-breaking has been suggested and used in practice to deal with indifferences in object allocation problems. Under general indifferences, TTC with fixed tie-breaking is neither Pareto efficient nor group strategy-proof. Furthermore, it may not select an allocation in the core of the market, even when the core is non-empty. We introduce a new setting, objective indifferences, in which any indifferences are shared by all agents. In this setting, which includes strict preferences as a special case, TTC with fixed tie-breaking maintains Pareto efficiency, group strategy-proofness, and core selection. Further, we characterize objective indifferences as the most general setting where TTC with fixed tie-breaking maintains these important properties.

1 Introduction

Important markets including living donor organ transplants, public housing assignments, and school choice can be modeled as Shapley-Scarf markets: each agent is endowed with an indivisible object and has preferences over the set of objects. Monetary transfers are disallowed, and participants have property rights to their own endowments. The goal is to re-allocate these objects among the agents to achieve efficiency and

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stability. The usual stability notion is the core: an allocation is in the core if no subset of agents would prefer to trade their endowments among themselves. In the original setting of Shapley and Scarf (1974), agents have strict preferences over the houses, and Gale's *top trading cycles* (TTC) algorithm finds an allocation in the core. Roth and Postlewaite (1977) further show that the core is non-empty, unique, and Pareto efficient. Roth (1982) shows that TTC is strategy-proof; Bird (1984), Moulin (1995), Pápai (2000), and Sandholtz and Tai (2024) show that it is group strategy-proof. These properties make TTC an attractive algorithm for practical applications.

However, the assumption that preferences are strict is quite strong. In particular, if any objects are essentially identical, agents should naturally be indifferent between them. For example, consider the problem of assigning students to college dormitory units. It seems reasonable to assume that two units with the same floor plan in the same building are basically equivalent from a student's perspective. In fact, the undergraduate on-campus housing application process at UC Berkeley applies this same logic. For first-year undergraduates, there are seven possible housing complexes (Unit 1, Unit 2, Unit 3, Stern, Foothill, Clark Kerr, and Blackwell), each with a variety of possible room configurations (double-occupancy room, triple-occupancy room, double-occupancy room in a 4-person suite, etc.). On housing applications, incoming first-year students rank their five most preferred *housing complex* × *floor plan* pairs. Partly, this is due to the practical challenges associated with collecting and aggregating students' preferences over the thousands of available dormitory units. More importantly, it demonstrates how it is implicitly assumed that students have strict preferences over *housing complex* × *floor plan* pairs, but are indifferent between dormitory units of the same type. Beyond college dormitory assignment problems, there are a host of real-world object assignment problems (military occupational specialty (MOS) assignment, school choice, etc.) that can be modeled with a similar structure.

We therefore present a model of Shapley-Scarf markets where there may be indistinguishable copies (which we will call "houses") of the objects (which we will call "types" or "house types"). Our model restricts agents to be indifferent between houses of the same type, but never indifferent between houses of different types. We call these preferences "objective indifferences." We see objective indifferences as a minimal model of indifferences, capturing the most basic and plausible form of indifferences.

In the fully general setting where agents' preferences may contain indifferences, *TTC with fixed tie-breaking* is often used in practice; ties in preference rankings are broken by some external rule. For example, Abdulkadiroglu and Sönmez (2003) propose something similar in the setting of school choice with priorities. However, TTC with fixed tie-breaking is not Pareto efficient nor group strategy-proof. In fact, there is an inherent tension between these two properties: Ehlers (2002) shows that when agents have weak preferences, there does not exist a Pareto efficient and group strategy-proof mechanism in Shapley-Scarf markets. With weak preferences, the core of the market may be empty or non-unique. But even when the core of a market is non-empty, TTC with fixed tie-breaking may not select a core allocation.

Objective indifferences adds structure to the case of general indifferences by constraining any indifferences to be universal among agents. While the core may still be empty, it is essentially single-valued when it is non-empty. That is, for any allocations in the core, all agents are indifferent between their allocated objects. In our setting, this means that every core allocation assigns an agent to the same house type, though not necessarily to the exact same house. Therefore, under objective indifferences, the core can be thought of as a unique mapping from agents to house types. Moreover, we show that in Shapley-Scarf markets with objective indifferences, TTC with fixed tie-breaking recovers Pareto efficiency and group strategy-proofness. It also selects a core allocation when the core is non-empty, and selects an allocation in the weak core otherwise. In fact, the objective indifferences setting is the most general setting such that TTC with fixed tie-breaking maintains any of these properties.

Others have have studied Shapley-Scarf markets with indifferences. Ehlers (2014) shows that in the general indifferences setting, a mechanism is individually rational, strategy-proof, weakly efficient, non-bossy, and consistent mechanism if and only if it is TTC for some fixed tie-breaking rule. Alcalde-Unzu and Molis (2011) and Jaramillo and Manjunath (2012) propose new families of trading cycle mechanisms, Top Trading Absorbing Sets (TTAS) and Top Cycle Rules (TCR), respectively, for the general indifferences setting. Both TTAS and TCR mechanisms are strategy-proof, Pareto efficient, and core selecting. Aziz and de Keijzer (2012) develop an even more general family of mechanisms, Generalized Absorbing Top Trading Cycles (GATTC), containing both TTAS and TCR as subclasses. GATTC mechanisms are Pareto efficient and core selecting, but are not generally strategy-proof. Plaxton (2013) defines a new subclass of GATTC mechanisms which are strategy-proof, Pareto efficient, core selecting, and run in $O(n^3)$-time, a substantial improvement over TCR mechansims, which run in $O(n^6)$-time, and TTAS mechanisms, which do not run in polynomial time. Fundamentally, the challenge for any mechanism in a Shapley-Scarf market with indifferences is determining which trading cycles to execute from among the many potential trading cycles that indifferences may induce. Using a fixed tie-breaking rule is intuitive and easy to implement, but as Ehlers (2014) shows, it comes at the expense of certain desirable properties, including Pareto efficiency and core selection. Though we too study Shapley-Scarf markets with indifferences, we place additional structure on the agents' indifferences and demonstrate how this resolves many of the challenges that indifferences pose.

Our paper makes several important new contributions to the literature on Shapley-Scarf markets. First, it defines and explores a new domain of preferences that accurately captures many real-world scenarios where this model is applied. Second, it characterizes the most general setting where TTC has no obvious drawbacks, in the sense that it retains all of the properties that make it so appealing under strict preferences. Third, it illustrates the underlying reason why weak preferences cause TTC to lose these properties: it is not indifferences per se, but *subjective* indifferences that may differ across agents.

Section 2 presents the formal notation. Section 3 explains TTC with fixed tie-breaking. Section 4 provides the main results. Section 5 concludes. Proofs of our results can be found in Appendix A.

2 Model

We present the model primitives. First we recount the classical Shapley and Scarf (1974) domain. Afterwards we introduce our "objective indifferences" domain.

We now present the general model of a Shapley-Scarf market. Let $N = \{1, \ldots, n\}$ be a finite set of agents, with generic member i. Let $H = \{h_1, \ldots, h_n\}$ be a set of houses, with generic member h. Every agent is endowed with one object, given by a bijection $w : N \to H$. The set of all endowments is $W(N, H)$, or W for short. An allocation is an assignment of an object to each agent, given by a bijection $x : N \to H$. The set of all allocations is $X(N, H)$, or simply X. For any $i \in N$, we use w_i and x_i as shorthand notation for $w(i)$ and $x(i)$ respectively. Similarly, for any $Q \subseteq N$ we use w_Q and x_Q as shorthand notation for $w(Q) = \{w(i) : i \in Q\}$ and $x(Q) = \{x(i) : i \in Q\}$ respectively.

A set $\mathcal{R}^n$ of possible preference profiles is a **domain**. Note that we restrict attention in this paper to domains that can be expressed as $\mathcal{R}^n$ for some set of preference relations $\mathcal{R}$ over H. That is, every agent has the same set of possible preference relations. If $\mathcal{R}$ is the set of strict preference relations over H, then $\mathcal{R}^n$ is the classical **strict preferences domain**. If $\mathcal{R}$ is the set of weak preference relations over H, then $\mathcal{R}^n$ is the classical **general indifferences domain**.

Our main domain is objective indifferences. Let $\mathcal{H} = \{H_1, H_2, \ldots, H_K\}$ be a partition of H. An element H_k of a partition is a **block**. Given H and $\mathcal{H}$, let $\eta : H \to \mathcal{H}$ be the mapping from a house to the partition element containing it; that is, $\eta(h) = H_k$ if $h \in H_k$. From any strict linear order $>$ over $\mathcal{H}$, we derive a preference relation $R_>$ over H: for $h, h' \in H$, we say that $hR_>h'$ if $\eta(h) > \eta(h')$ or $\eta(h) = \eta(h')$.

Let $\mathcal{R}(\mathcal{H}) := \{R_>\}_>$ be the set of all $R_>$ given $\mathcal{H}$. Given $\mathcal{H}$, $\mathcal{R}(\mathcal{H})^n$ is an **objective indifferences domain**. We sometimes suppress $(\mathcal{H})$ from the notation when context makes it clear. Note that all agents are indifferent between houses in the same block of $\mathcal{H}$ and have strict preferences between houses in different blocks. Thus, we may refer to the "indifference classes" for the domain with the understanding that everyone shares the same indifference classes. As usual, we let P_i and I_i denote the strict relation and indifference relation associated with a preference relation $R_i \in \mathcal{R}(\mathcal{H})$. For any $Q \subseteq N$ and preference profile R, we use R_Q to denote $\{R_i : i \in Q\}$.

2.1 Mechanisms

This subsection recounts formalities on mechanisms and top trading cycles. Familiar readers may safely skip this subsection.

A **market** is a tuple (N, H, w, R). A **mechanism** is a function $f : \mathcal{R}^n \to X$; given a preference profile, it produces an allocation. When it is unimportant or clear from context, we suppress inputs from the notation. For any $i \in N$, let $f_i(R)$ denote i's allocated house under $f(R)$. Similarly, for any $Q \subseteq N$, let $f_Q(R) = \{f_i(R) : i \in Q\}$. Fix a mechanism f, a market (N, H, w), and a preference domain $\mathcal{R}^n$. We work with the following axioms.

A mechanism is Pareto efficient if it always produces Pareto efficient allocations.

Pareto efficiency (PE). For all $R \in \mathcal{R}^n$, there is no other allocation $x \in X$ such that $x_i R_i f_i(R)$ for all $i \in N$ and $x_i P_i f_i(R)$ for at least one $i \in N$.

Group strategy-proofness requires that no coalition of agents can collectively improve their outcomes by submitting false preferences. Note that in the following definition, we require both the true preferences and misreported preferences to come from the preference domain $\mathcal{R}^n$.

Group strategy-proofness (GSP). For all $R \in \mathcal{R}^n$, there do not exist $Q \subseteq N$ and $R' = (R'_Q, R_{-Q}) \in \mathcal{R}^n$ such that $f_i(R') R_i f_i(R)$ for all $i \in Q$ with $f_i(R') P_i f_i(R)$ for at least one $i \in Q$.

Individual rationality models the constraint of voluntary participation. It requires that agents receive a house they weakly prefer to their endowment.

Individual rationality (IR). For all $R \in \mathcal{R}^n$, $f_i(R) R_i w_i$ for all $i \in N$.

We also define the core of a market: an allocation is in the core if there is no subset of agents who could benefit from trading their endowments among themselves.

Definition 1. An allocation x is **blocked** if there exists a coalition $Q \subseteq N$ and allocation x' such that $x'_Q = w_Q$ and $x'_i R_i x_i$ for all $i \in Q$, with $x'_i P_i x_i$ for at least one $i \in Q$.

Definition 2. An allocation x is in the **core** of the market if it is not blocked.

The core property models the restriction imposed by property rights. Notice that individual rationality excludes blocking coalitions of size 1. The last axiom is core-selecting.

Core-selecting (CS). For all $R \in \mathcal{R}$, if the core of the market is non-empty then $f(R)$ is in the core.

In Section 4, we characterize maximal domains on which TTC with fixed tie-breaking satisfies the axioms. By a "maximal" domain, we mean the following.

Definition 3. A domain $\mathcal{R}^n$ is **maximal** for an axiom A and a class of rules F if

1. each $f \in F$ is A on $\mathcal{R}^n$, and
2. for any $\tilde{\mathcal{R}}^n \supset \mathcal{R}^n$, there is some $f \in F$ that is *not* A on $\tilde{\mathcal{R}}^n$.

Note that this definition of maximality depends on both the axiom and the class of rules, which differs from elsewhere in the literature. Typically, a maximal domain for some property is the largest possible domain on which *some* rule exists which satisfies the desired property. We focus on a specific class of rules: top trading cycles with fixed tie-breaking. Again note that we only consider domains that can be written as $\mathcal{R}^n$, which is common.

3 Top trading cycles with fixed tie-breaking

In this paper, we analyze top trading cycles (TTC) with fixed tie-breaking on the domains defined in the previous section. For an extensive history of TTC, we refer the reader to Morill and Roth (2024). We briefly define TTC and TTC with fixed tie-breaking.

Algorithm 1. ***Top Trading Cycles***. *Consider a market (N, H, w, R) under strict preferences. Draw a graph with N as nodes.*

1. *Draw an arrow from each agent i to the owner (endowee) of his favorite remaining object.*
2. *There must exist at least one cycle; select one of them. For each agent in this cycle, give him the object owned by the agent he is pointing at. Remove these agents from the graph.*
3. *If there are remaining agents, repeat from step 1.*

We denote the resulting allocation as $\text{TTC}(R)$.

TTC is well-defined only with strict preferences, as Step 1 requires a unique favorite object. In practice, a **fixed tie-breaking profile** $\succ$ is often used to resolve indifferences. Given N, let $\succ = (\succ_1, \ldots, \succ_n)$, where each $\succ_i$ is a strict linear order over N. This linear order will be used to break indifferences between objects (based on their owners). For any preference relation R_i and tie-breaking rule $\succ_i$, let $R_{i,\succ_i}$ be given by the following. For any $j \neq j'$, let $w_j R_{i,\succ_i} w_{j'}$ if

1. $w_j P_i w_{j'}$, or
2. $w_j I_i w_{j'}$ and $j \succ_i j'$

Then $R_{i,\succ_i}$ is a strict linear order over the individual houses. Example 1 illustrates how we combine an agent's preferences and the tie-breaking rule to construct tie-broken preferences.

Example 1. Let $N = \{1, 2, 3, 4\}$. Agent 1's preferences R_1 and tie-breaking rule $\succ_1$ are shown below. In our visual representations of preference relations, each line represents an indifference class, and the houses on any line are strictly preferred to houses on lines below them. For example, the representation of R_1 below indicates that $w_3 \; I_1 \; w_4 \; P_1 \; w_1 \; I_1 \; w_2$.

R_1		$\succ_1$		$R_{1,\succ_1}$
w_3, w_4		1		w_3
w_1, w_2	+	2	$\rightarrow$	w_4
		3		w_1
		4		w_2

Since $w_3 I_1 w_4$ and $3 \succ_1 4$, we have $w_3 P_{1,\succ_1} w_4$. Likewise, since $w_1 I_1 w_2$ and $1 \succ_1 2$, we have $w_1 P_{1,\succ_1} w_2$. Therefore, agent 1's complete tie-broken preferences $R_{1,\succ_1}$ are given by $w_3 \; P_{1,\succ_1} \; w_4 \; P_{1,\succ_1} \; w_1 \; P_{1,\succ_1} \; w_2$.

Given a preference profile $R \in \mathcal{R}^n$ and a tie-breaking profile $\succ$, let $R_\succ = (R_{1,\succ_1}, \ldots, R_{n,\succ_n})$. **TTC with fixed tie-breaking (TTC$_\succ$)** is $\text{TTC}_\succ(R) \equiv \text{TTC}(R_\succ)$. That is, the tie-breaking profile is used to generate strict preferences, and TTC is applied to the resulting strict preference profile. Formally, each tie-breaking profile $\succ$ generates a different TTC$_\succ$ rule. For a given R and $\succ$, we use $TTC_\succ(R)$ to refer both to the step-by-step procedure of TTC$_\succ$, and to the final allocation it generates. The following example illustrates how TTC$_\succ$ works in the objective indifferences domain.

Example 2. Let $N = \{1, 2, 3, 4\}$. The preference profile $R = (R_1, R_2, R_3, R_4)$ and tie-breaking profile $\succ = (\succ_1, \succ_2, \succ_3, \succ_4)$ are shown below. R and $\succ$ are combined as shown in Example 1 to construct the tie-broken preference profile $R_\succ = (R_{1,\succ_1}, R_{2,\succ_2}, R_{3,\succ_3}, R_{4,\succ_4})$. Recall that $\text{TTC}_\succ(R)$ is equivalent to $\text{TTC}(R_\succ)$.

R_1	R_2	R_3	R_4
w_2, w_3	w_1	w_1	w_2, w_3
w_1	w_2, w_3	w_4	w_4
w_4	w_4	w_2, w_3	w_1

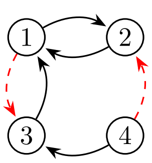
Preference profile R

$\succ_1$	$\succ_2$	$\succ_3$	$\succ_4$
2	1	3	3
1	2	2	1
3	3	1	2
4	4	4	3

Tie-breaking profile $\succ$

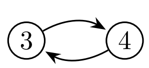
$R_{1,\succ_1}$	$R_{2,\succ_2}$	$R_{3,\succ_3}$	$R_{4,\succ_4}$
w_2	w_1	w_1	w_3
w_3	w_2	w_4	w_2
w_1	w_3	w_3	w_4
w_4	w_4	w_2	w_1

Tie-broken preference profile $R_\succ$



Step 1 of $\text{TTC}_\succ(R)$:

Each agent points to the owner of their favorite house according to their *tie-broken* preferences, represented by the black arrows. The red dashed arrows are only shown to emphasize that agents 1 and 4 are indifferent between their top choices, w_2 and w_3. Agents 1 and 2 form a cycle, and therefore swap houses.



Step 2 of $\text{TTC}_\succ(R)$:

After removing the agents assigned in Step 1, the remaining agents (3 and 4) point to the owner of their favorite remaining house. They form a cycle, and therefore swap houses. Since every agent has been assigned to a house, the TTC$_\succ$ procedure ends. The resulting allocation is $x = (w_2, w_1, w_4, w_3)$.

4 Results

In the general indifferences domain, TTC$_\succ$ mechanisms are not Pareto efficient, core-selecting, nor group strategy-proof. We give some simple examples below to illustrate these failures. However, we show that in the objective indifferences domain, TTC$_\succ$ mechanisms satisfy all three properties. Furthermore, we show that objective indifferences characterizes the set of maximal domains on which TTC$_\succ$ mechanisms are PE and CS, and characterizes the set of "symmetric-maximal" domains on which $\text{TTC}_{\succ}$ mechanisms are GSP.

4.1 Pareto efficiency and core-selecting

When we relax the assumption of strict preferences and allow for general indifferences, $\text{TTC}_{\succ}$ loses two of its most appealing properties: Pareto efficiency and core-selecting. However, in the intermediate case of objective indifferences, $\text{TTC}_{\succ}$ retains these two properties. Moreover, on *any* larger domain, $\text{TTC}_{\succ}$ loses both Pareto efficiency and core-selecting. Thus, we show that it is not indifferences per se, but rather *subjective* evaluations of indifferences which cause $\text{TTC}_{\succ}$ to lose these properties.

We first demonstrate that $\text{TTC}_{\succ}$ mechanisms are not Pareto efficient under general indifferences. Example 3 gives the simplest case.

Example 3. Let $N = \{1, 2\}$. The preference profile $R = (R_1, R_2)$, tie-breaking profile $\succ = (\succ_1, \succ_2)$, and tie-broken preference profile $R_{\succ} = (R_{1,\succ_1}, R_{2,\succ_2})$ are shown below.

R_1	R_2
w_1, w_2	w_1
	w_2

Preference profile R

$\succ_1 = \succ_2$
1
2

Tie-breaking profile $\succ$

$R_{1,\succ_1}$	$R_{2,\succ_2}$
w_1	w_1
w_2	w_2

Tie-broken preference profile $R_{\succ}$

The $\text{TTC}_{\succ}$ allocation is $x = (w_1, w_2)$, which is Pareto dominated by $x' = (w_2, w_1)$ since

$$(x'_1 =)\ w_2 I_1 w_1\ (= x_1) \quad \text{and} \quad (x'_2 =)\ w_1 P_2 w_2\ (= x_2).$$

This example demonstrates the underlying reason that $\text{TTC}_{\succ}$ fails PE under general indifferences: tie-breaking rules may not take advantage of Pareto gains made possible by the agents' indifferences. However, under objective indifferences, if any agent is indifferent between two houses, then all agents are indifferent between those two houses. Consequently, objective indifferences rules out situations like the one shown in Example 3.

Under general indifferences, the set of core allocations may not be a singleton; there may be no core allocations or there may be multiple. As Example 3 demonstrates, even when the core of the market is non-empty, $\text{TTC}_{\succ}$ may still fail to select a core allocation.[1] However, under objective indifferences, if the core of a market is non-empty then $\text{TTC}_{\succ}$ mechanisms always select a core allocation.

In fact, the objective indifferences setting characterizes the entire set of maximal domains on which $\text{TTC}_{\succ}$ mechanisms are Pareto efficient or core-selecting. That is, if all $\text{TTC}_{\succ}$ mechanisms are PE or CS on a domain $\mathcal{R}^n$, then it must be a weak subset of some objective indifferences domain. Conversely, for any superset of an objective indifferences domain, there is some $\text{TTC}_{\succ}$ mechanism that is not PE or CS.

[1] It is straightforward to see that $x' = (w_2, w_1)$ is in the core of the market and $x = (w_1, w_2)$ is not.

Theorem 1. *The following are equivalent:*

1. *$\mathcal{R}^n$ is an objective indifferences domain.*
2. *$\mathcal{R}^n$ is a maximal domain on which $TTC_{\succ}$ mechanisms are Pareto efficient.*
3. *$\mathcal{R}^n$ is a maximal domain on which $TTC_{\succ}$ mechanisms are core-selecting.*

Proof. Appendix A.1. □

The full proof is in the appendix, but the intuition is simple. The objective indifferences domain precludes possibilities such as Example 3, and any larger domain inevitably introduces the possibility of such a pair.

It follows from Sönmez (1999) that under objective indifferences, the core of a market is *essentially single-valued* when it exists. That is, for any two allocations x and y in the core of a market, we have $x_i I_i y_i$ for all agents i. In our proof of Theorem 1, we also prove this claim directly. Since the core is essentially single-valued, under objective indifferences the core can be thought of as a unique mapping from agents to house types. In other words, the core allocations are permutations of one another where agents may be assigned to different houses, but always receive houses of the same type.

Corollary 1. *For any two allocations $x \neq y$ in the core of an objective indifferences market, $x_i I_i y_i$ for all $i \in N$.*

Proof. Appendix A.1. □

Though all $\text{TTC}_{\succ}$ mechanisms are core-selecting under objective indifferences, the core of the market may be empty, as the following simple example shows.

Example 4. Let $N = \{1, 2, 3\}$. It is easy to verify that for the preference profile $R = (R_1, R_2, R_3)$ shown below, there are no core allocations.

R_1	R_2	R_3
w_2, w_3	w_1	w_1
w_1	w_2, w_3	w_2, w_3

Any allocation x such that $x_1 \in \{w_1, w_2\}$ is blocked by $Q = \{1, 3\}$ and $x' = (w_3, w_1)$. Similarly, any allocation x such that $x_1 = w_3$ is blocked by $Q = \{1, 2\}$ and $x' = (w_2, w_1)$.

When the core of an objective indifferences market is empty, all $\text{TTC}_{\succ}$ mechanisms select an allocation in the **weak core** of the market. In fact, even under general indifferences, the weak core is non-empty and $\text{TTC}_{\succ}$ mechanisms select an allocation in the weak core.

Definition 4. An allocation x is **weakly blocked** if there exists a coalition $Q \subseteq N$ and allocation x' such that $x'_Q = w_Q$ and $x'_i P_i x_i$ for all $i \in Q$.

Definition 5. An allocation x is in the **weak core** of the market if it is not weakly blocked.

Proposition 1. *For any market, the weak core is non-empty and* $TTC_{\succ}$ *mechanisms select an allocation in the weak core.*

Proof. Appendix A.1. □

4.2 Group strategy-proofness

$TTC_{\succ}$ also loses group strategy-proofness once we move from strict preferences to weak preferences. However, in the intermediate case of objective indifferences, $TTC_{\succ}$ recovers group strategy-proofness. Further, $TTC_{\succ}$ mechanisms are not GSP in any larger "symmetric" domain. We say that a domain is symmetric if, when $h_1 P_i h_2$ is possible, then so is $h_2 P_i h_1$. We will informally argue that this is not an onerous modeling restriction.

First we present a simple example demonstrating that under general indifferences, $TTC_{\succ}$ mechanisms are not group strategy-proof. Example 5 shows how an agent can break his own indifference to benefit a coalition member without harming himself.

Example 5. Let $N = \{1, 2, 3\}$ and let $Q = \{1, 3\}$. For the preference profile $R = (R_1, R_2, R_3)$ and tie-breaking profile $\succ = (\succ_1, \succ_2, \succ_3)$ shown below, the $TTC_{\succ}$ allocation is $x = (w_2, w_1, w_3)$. However, if agent 1 were to report R'_1, then for $R' = (R'_1, R_2, R_3)$ the $TTC_{\succ}$ allocation is $x' = (w_3, w_2, w_1)$. Note that $x'_3 P_3 x_3$ and $x'_1 I_1 x_1$, so $TTC_{\succ}$ is not GSP.

R_1	R_2	R_3
w_2, w_3	w_1	w_1
w_1	w_2	w_2
	w_3	w_3

Preference profile R

R'_1	R_2	R_3
w_3	w_1	w_1
w_2	w_2	w_2
w_1	w_3	w_3

Preference profile R'

$\succ_1 = \succ_2 = \succ_3$
1
2
3

Tie-breaking profile $\succ$

Objective indifferences excludes situations like Example 5 in two ways. First, it eliminates the possibility that one agent is indifferent between two houses while another agent has a strict preference. Second, it constrains the possible set of misreports available to a manipulating coalition, since agents can *only* report indifference among all houses of the same type.[2] Our next result characterizes the set of symmetric-maximal domains on which $TTC_{\succ}$ mechanisms are GSP.

Before presenting our result, we must define "symmetric" and "symmetric-maximal" domains.

Definition 6. A domain $\mathcal{R}^n$ is **symmetric** if for any $h_1, h_2 \in H$, if there exists $R_i \in \mathcal{R}$ such that $h_1 P_i h_2$, then there also exists $R'_i \in \mathcal{R}$ such that $h_2 P'_i h_1$.

Definition 7. A domain $\mathcal{R}^n$ is **symmetric-maximal** for an axiom A and a class of rules F if

1. $\mathcal{R}^n$ is symmetric,

[2] The constraint on agents' reports is an important difference from Ehlers (2002).

2. each $f \in F$ is A on $\mathcal{R}^n$, and

3. for any symmetric $\tilde{\mathcal{R}}^n \supset \mathcal{R}^n$, there is some $f \in F$ that is *not* A on $\tilde{\mathcal{R}}^n$.

In practical applications, symmetry is a natural restriction to place on the domain; if it is possible that agents might report strictly preferring some house h to another house h', we should not preclude the possibility they strictly prefer h' to h. Indeed, the point of mechanism design is that preferences are unknown and must be solicited. It is easy to see that objective indifferences domains are symmetric. Relative to maximality, symmetric-maximality restricts the possible expansions of objective indifferences domains that we must consider.

Theorem 2. *$\mathcal{R}^n$ is a symmetric-maximal domain on which $TTC_\succ$ mechanisms are group strategy-proof if and only if it is an objective indifferences domain.*

Proof. Appendix A.2. □

Our proof uses similar reasoning to the proof that TTC is group strategy-proof under strict preferences contained in Sandholtz and Tai (2024). Any coalition requires a "first mover" to misreport, but this agent must receive an inferior house to the one he originally received. In the following example, we note that objective indifferences domains are *not* maximal domains on which $\text{TTC}_\succ$ mechanisms are GSP.

Example 6. Let $N = \{1, 2\}$, $H = \{h_1, h_2\}$, and $\mathcal{H} = \{\{h_1, h_2\}\}$. Suppose $\mathcal{R}' = \mathcal{R}(\mathcal{H}) \cup (h_1 P h_2) = \{(h_1 I h_2), (h_1 P h_2)\}$. That is, expand the objective indifferences domain induced by $\mathcal{H}$ to include the ordering $(h_1 P h_2)$. Note that this expanded domain is not symmetric, since $\mathcal{R}'$ does not also contain the preference ordering $(h_2 P h_1)$.

Let $\succ = ((1 \succ_1 2), (1 \succ_2 2))$. We will show that for any market (N, H, w, R), $\text{TTC}_\succ$ is group strategy-proof. It is straightforward to show that the same is true for the remaining 3 possible tie-breaking profiles.

Without loss of generality, assume $w_i = h_i$. If both agents have the same preferences, then there is clearly no profitable group manipulation. Consider the following two possible preference profiles:

R_1	R_2
h_1	h_1, h_2
h_2	

or

R_1	R_2
h_1, h_2	h_1
	h_2

In the first case, the $\text{TTC}_\succ$ allocation is $x = (h_1, h_2)$, so both agents receive one of their most preferred houses. Therefore, it is not possible for either agent to strictly improve. In the second case, the $\text{TTC}_\succ$ allocation is $x = (h_1, h_2)$. It would benefit agent 2 for agent 1 to rank h_2 above h_1, since agent 1 is indifferent between h_1 and h_2. However, this is not possible since $(h_2 P h_1) \notin \mathcal{R}'$.

5 Conclusion

Our main set of results show that objective indifferences domains are maximal domains on which $\text{TTC}_{\succ}$ mechanisms are Pareto efficient, core selecting, and group strategy-proof. It is remarkable that the maximal domains on which $\text{TTC}_{\succ}$ satisfies these three distinct properties (essentially) coincide. We therefore view objective indifferences domains as the most general possible setting where $\text{TTC}_{\succ}$ can be applied without any tradeoffs. Moreover, we interpret our results as showing that it *subjective* indifferences, not indifferences themselves, which cause issues for TTC when we relax the assumption of strict preferences.

Therefore, in markets where one could reasonably assume that any indifferences are shared among all agents, $\text{TTC}_{\succ}$ is a sensible choice of mechanism. Even when the market imposes constraints on the possible tie-breaking rules, it is guaranteed that $\text{TTC}_{\succ}$ will be PE, CS, and GSP regardless of which tie-breaking rule is chosen. Moreover, $\text{TTC}_{\succ}$ is computationally efficient, as well as easy to explain and implement.

We do not believe our results imply that $\text{TTC}_{\succ}$ should be avoided in settings beyond objective indifferences, or that market designers should only allow objective indifference preference reports. Consider school choice in San Francisco, which uses a lottery system to assign school seats at most public schools. While current details are not readily available, Abdulkadiroglu, Featherstone, Niederle, Pathak, and Roth designed a system using TTC.[3] At some schools, there are seats dedicated to language immersion programs and other seats that are intended for general education. For instance, at West Portal Elementary School, there are roughly 120 seats, approximately 25% of which are for Cantonese immersion.[4] The Cantonese immersion seats at West Portal are further divided into seats reserved for children who are already bilingual and students who do not yet speak Cantonese. Suppose families' preferences over schools can be described by objective indifferences. That is, suppose families care only about which school they attend, and are indifferent about what kind of seat they receive. Our results suggest that $\text{TTC}_{\succ}$ is an excellent candidate mechanism for this setting.

However, the real situation may be more complicated. Perhaps some families are indifferent between bilingual and regular seats, while other families have strict preferences for one type of seat or the other. For example, a family whose children are already bilingual in Cantonese may be indifferent between the two types of seats at West Portal, while another family may have a strict preference for cultural community through the Cantonese bilingual program. In this case, our results show that $\text{TTC}_{\succ}$ mechanisms are no longer PE, CS, nor GSP. However, the policy implications are not clear. While there are mechanisms for the general indifferences case, these mechanisms may have other tradeoffs, such as increased computational or cognitive complexity. Our results lay out exactly the situations where $\text{TTC}_{\succ}$ is Pareto efficient, core-selecting, and group strategy-proof. However, the results do not necessarily proscribe its use outside of these settings. Rather, one could view this set of results as rationalizing the use of $\text{TTC}_{\succ}$ in many settings where $\text{TTC}_{\succ}$

[3]See the blog post by Al Roth: https://marketdesigner.blogspot.com/2010/09/san-francisco-school-choice-goes-in.html. As he notes, the team were not privy to the implementation or resulting data.

[4]https://web.archive.org/web/20250422170224/https://www.sfchronicle.com/bayarea/article/sfusd-competitive-public-schools-20252957.php

has actually been suggested and applied.

Our paper opens interesting new lines of inquiry. First, we believe that studying matching markets with constrained indifferences is an exciting avenue for future research. In many real-world matching markets, agents have indifferences, but often with a certain structure imposed by the specific market. Understanding how adding structure to the case of general indifferences may affect matching problems is not only theoretically interesting, but could improve policy choices. For instance, there may be tradeoffs in the selection of the partition $\mathcal{H}$ given the set of objects H. In some cases, there may be some ambiguity: are two dorms with the same floor plan but on different floors of the same building equivalent? Inappropriately combining indifference classes might lead to efficiency losses in the spirit of Example 3. On the other hand, splitting indifference classes might allow group manipulations like in Example 5. We leave formal results as future work. We also leave an axiomatic characterization of $\text{TTC}_{\succ}$ on objective indifferences domains as future work.

References

Abdulkadiroglu, A., & Sönmez, T. (2003). School choice: A mechanism design approach. *American Economic Review*, *93*(3), 729–747.

Alcalde-Unzu, J., & Molis, E. (2011). Exchange of indivisible goods and indifferences: The Top Trading Absorbing Sets mechanisms. *Games and Economic Behavior*, *73*(1), 1–16.

Aziz, H., & de Keijzer, B. (2012). Housing markets with indifferences: A tale of two mechanisms. *Proceedings of the Twenty-Sixth AAAI Conference on Artificial Intelligence*.

Bird, C. G. (1984). Group incentive compatibility in a market with indivisible goods. *Economics Letters*, *14*.

Ehlers, L. (2002). Coalitional strategy-proof house allocation. *Journal of Economic Theory*, *105*(2), 298–317.

Ehlers, L. (2014). Top trading with fixed tie-breaking in markets with indivisible goods. *Journal of Economic Theory*, *151*, 64–87.

Jaramillo, P., & Manjunath, V. (2012). The difference indifference makes in strategy-proof allocation of objects. *Journal of Economic Theory*, *147*(5), 1913–1946.

Morill, T., & Roth, A. E. (2024). Top trading cycles. *Journal of Mathematical Economics*, *112*.

Moulin, H. (1995). *Cooperative microeconomics: A game-theoretic introduction*. Prineton University Press.

Pápai, S. (2000). Strategyproof assignment by hierarchical exchange. *Econometrica*, *68*.

Plaxton, C. G. (2013). A simple family of top trading cycles mechanisms for housing markets with indifferences. *Proceedings of the 24th International Conference on Game Theory*.

Roth, A. E. (1982). Incentive compatibility in a market with indivisible goods. *Economics Letters*, *9*(2), 127–132.

Roth, A. E., & Postlewaite, A. (1977). Weak versus strong domination in a market with indivisible goods. *Journal of Mathematical Economics*, *4*(2), 131–137.

Sandholtz, W., & Tai, A. (2024). Group incentive compatibility in a market with indivisible goods: A comment. *Economics Letters*, *243*.

Shapley, L., & Scarf, H. (1974). On cores and indivisibility. *Journal of Mathematical Economics*, *1*(1), 23–37.

Sönmez, T. (1999). Strategy-proofness and essentially single-valued cores. *Econometrica*, *67*(3), 677–689.

Takamiya, K. (2001). Coalition strategy-proofness and monotonicity in shapley–scarf housing markets. *Mathematical Social Sciences*, *41*.

Appendix A Proofs

We provide proofs for the results in the main text. Note that individual rationality (IR) of TTC$_\succ$ follows immediately from IR of TTC and the fact that $TTC_\succ(R) \equiv TTC(R_\succ)$.

Given a market (N, H, w, R) and mechanism TTC$_\succ$, let $S_k(R)$ be the kth cycle executed in the process of TTC$_\succ(R)$. We will appeal to the following fact.

Fact 1. *Fix a market (N, H, w, R) and a tie-breaking profile $\succ$. Let $x = TTC_\succ(R)$. If $i \in S_k(R)$ and $x_j P_i x_i$, then $j \in \cup_{\ell=1}^{k-1} S_\ell(R)$.*

Fact 1 follows from the definitions: $x_j P_i x_i$ implies $x_j P_{i,\succ_i} x_i$, and TTC$_\succ(R) \equiv$ TTC$(R_\succ)$. Under TTC$(R_\succ)$, if $x_j P_{i,\succ_i} x_i$, then j must have been assigned before i; otherwise at step k, i would have pointed at x_j's owner instead of at x_i's owner.

We also make use of the following lemma.

Lemma 1. *Fix (N, H, w) and a domain $\mathcal{R}^n$. For any two preference relations $R_\alpha, R_\beta \in \mathcal{R}$ and houses $h_1, h_2 \in H$, let*

$$\{i \in N : w_i P_\alpha h_1\} \subseteq A \subseteq \{i \in N : w_i R_\alpha h_1\}$$

and let

$$\{i \in A^c : w_i P_\beta h_2\} \subseteq B \subseteq \{i \in A^c : w_i R_\beta h_2\}.$$

Let $\succ$ be any tie-breaking profile such that for all $i \in N$, $i \succ_i j$ for all $j \neq i$. Fix a preference profile $R \in \mathcal{R}^n$ and let $x = TTC_\succ(R)$. If $R_i = R_\alpha$ for all $i \in A$ and $R_i = R_\beta$ for all $i \in B$, then $x_i = w_i$ for all $i \in A \cup B$.

Proof of Lemma 1. First we show that $x_i = w_i$ for all $i \in A$. Toward a contradiction, suppose that $W = \{i \in A : x_i \neq w_i\}$ is non-empty. Take some agent $i^* \in W$ such that $w_{i^*} R_\alpha w_i$ for all $i \in W$. By individual rationality of x, $x_{i^*} R_{i^*} w_{i^*}$. Also, since $i^* \succ_{i^*} i$ for all $i \neq i^*$, by definition of $TTC_\succ$ we know that if $x_{i^*} I_{i^*} w_{i^*}$ then $x_{i^*} = w_{i^*}$. Therefore, since we assumed that $x_{i^*} \neq w_{i^*}$, $x_{i^*} R_{i^*} w_{i^*}$ implies $x_{i^*} P_{i^*} w_{i^*}$. Since $i^* \in W \subseteq A$, $R_{i^*} = R_\alpha$; therefore, $x_{i^*} P_{i^*} w_{i^*}$ implies $x_{i^*} P_\alpha w_{i^*}$. Consider the agent $j \in N$ such that $x_{i^*} = w_j$. Note that $j \in A$, since $i^* \in A$ and $w_j P_\alpha w_{i^*}$. Also, $x_j \neq w_j$, so $j \in W$. But this contradicts our assumption that $w_{i^*} R_\alpha w_i$ for all $i \in W$.

Next we show that $x_i = w_i$ for all $i \in B$. Toward a contradiction, suppose that $W = \{i \in B : x_i \neq w_i\}$ is non-empty. Take some $i^* \in W$ such that $w_{i^*} R_\beta w_i$ for all $i \in W$. By individual rationality of x, $x_{i^*} R_{i^*} w_{i^*}$. Also, since $i^* \succ_{i^*} i$ for all $i \neq i^*$, by definition of $TTC_\succ$ we know that if $x_{i^*} I_{i^*} w_{i^*}$ then $x_{i^*} = w_{i^*}$. Therefore, since we assumed that $x_{i^*} \neq w_{i^*}$, $x_{i^*} R_{i^*} w_{i^*}$ implies $x_{i^*} P_{i^*} w_{i^*}$. Since $i^* \in W \subseteq B$, $R_{i^*} = R_\beta$; therefore, $x_{i^*} P_{i^*} w_{i^*}$ implies $x_{i^*} P_\beta w_{i^*}$. Consider the agent $j \in N$ such that $x_{i^*} = w_j$. We know that $j \in A^c$, because we showed that $x_i = w_i$ for all $i \in A$ and $x_j \neq w_j$. Therefore, $j \in B$, since $i^* \in B$ and $w_j P_\beta w_{i^*}$. Also, since $x_j \neq w_j$, $j \in W$. But this contradicts our assumption that $w_{i^*} R_\beta w_i$ for all $i \in W$. □

Appendix A.1 Pareto efficiency and core-selecting

Theorem 1. *The following are equivalent:*

1. *$\mathcal{R}^n$ is an objective indifferences domain.*

2. *$\mathcal{R}^n$ is a maximal domain on which $TTC_\succ$ mechanisms are Pareto efficient.*

3. *$\mathcal{R}^n$ is a maximal domain on which $TTC_\succ$ mechanisms are core-selecting.*

Fix (N, H, w). The result is trivial for $|N| = 1$, so assume $|N| \geq 2$.

Proof of 1. $\iff$ 2. First we show that for any objective indifferences domain, $\text{TTC}_\succ$ mechanisms are PE. Fix some tie-breaking rule $\succ$. Let $\mathcal{H}$ be any partition of H and let $R \in \mathcal{R}(\mathcal{H})^n$. If $\mathcal{H} = \{H\}$, the result is trivial, so suppose the partition has at least two blocks. Let $x = TTC_\succ(R)$, and suppose that some feasible allocation y Pareto dominates x. Let $W = \{i \in N : y_i P_i x_i\}$ be the set of agents who strictly improve under y, which must be non-empty. Let k be the first step in the process of $TTC_\succ(R)$ that an agent in W is assigned. That is, $\cup_{\ell=1}^{k-1} S_\ell(R) \cap W = \emptyset$ and $S_k(R) \cap W \neq \emptyset$. Take some $i^* \in S_k(R) \cap W$. If $y_{i^*} P_{i^*} x_{i^*}$, then by definition of objective indifferences, $\eta(y_{i^*}) \neq \eta(x_{i^*})$. Therefore, Fact 1 implies that $\{i \in N : x_i \in \eta(y_{i^*})\} \subseteq \cup_{\ell=1}^{k-1} S_\ell(R)$. By feasibility of y, if $\eta(y_{i^*}) \neq \eta(x_{i^*})$, there must be an agent $j \in \cup_{\ell=1}^{k-1} S_\ell(R)$ for whom $x_j \in \eta(y_{i^*})$ but $y_j \notin \eta(y_{i^*})$. Therefore, $\neg(y_j I_j x_j)$. Since y Pareto dominates x, it must be that $y_j P_j x_j$. But then $j \in W$, contradicting our assumption that $\cup_{\ell=1}^{k-1} S_\ell(R) \cap W = \emptyset$.

Next we show that for any domain $\tilde{\mathcal{R}}^n$ where $\tilde{\mathcal{R}}^n \nsubseteq \mathcal{R}(\mathcal{H})^n$ for any partition $\mathcal{H}$ of H, $\text{TTC}_\succ$ mechanisms are not PE on $\tilde{\mathcal{R}}^n$. If $\tilde{\mathcal{R}}^n \nsubseteq \mathcal{R}(\mathcal{H})^n$ for any $\mathcal{H}$, then $\tilde{\mathcal{R}}$ must contain two orderings, R_α and R_β, such that for some $h_1, h_2 \in H$ we have $h_1 I_\alpha h_2$ but $h_1 P_\beta h_2$. Taking only the existence of $R_\alpha, R_\beta \in \tilde{\mathcal{R}}$ for granted, we find a preference profile $R \in \tilde{\mathcal{R}}^n$ and tie-breaking profile $\succ$ such that $TTC_\succ(R)$ is not PE. Without loss of generality, assume $w_i = h_i$ for all $i \in N$. Define $A = \{i \in N : w_i R_\alpha w_1\} \setminus \{2\}$. Consider the preference profile R such that

$$R_i = \begin{cases} R_\alpha & \text{if } i \in A \\ R_\beta & \text{if } i \in A^c. \end{cases}$$

Note that $1 \in A$ and $2 \in A^c$, so $R_1 = R_\alpha$ and $R_2 = R_\beta$. Take any tie-breaking profile $\succ$ such that for all $i \in N$, $i \succ_i j$ for all $j \neq i$. Let $x = \text{TTC}_\succ(R)$. It follows directly from Lemma 1 that $x = w$. However, note that $w_2 I_1 w_1 (= x_1)$ and $w_1 P_2 w_2 (= x_2)$, so x is Pareto dominated by $y = (w_2, w_1, w_3, ..., w_n)$. □

Proof of (1) $\iff$ (3). First we show that for any objective indifferences domain, $\text{TTC}_\succ$ mechanisms are CS. Fix some tie-breaking rule $\succ$. Let $\mathcal{H}$ be any partition of H and let $R \in \mathcal{R}(\mathcal{H})^n$. If $\mathcal{H} = \{H\}$, the result is trivial, so suppose the partition has at least two blocks. Suppose that the core of (N, H, w, R) is non-empty and contains some allocation y. Let $x = TTC_{\succ}(R)$. It suffices to show that $x_i I_i y_i$ for all $i \in N$. We proceed by induction on the steps of $TTC_{\succ}(R)$.

> Step 1 By definition of TTC$_{\succ}$, for all $i \in S_1(R)$ we know that $x_i R_i h$ for all $h \in H$. Therefore, $x_i R_i y_i$ for all $i \in S_1(R)$. Suppose there is some $i^* \in S_1(R)$ such that $x_{i^*} P_{i^*} y_{i^*}$. Then $S_1(R)$ and x block y, contradicting our assumption that y is in the core. Thus, $x_i I_i y_i$ for all $i \in S_1(R)$.
>
> Step k Assume that $x_i I_i y_i$ for all $i \in \cup_{\ell=1}^{k-1} S_\ell(R)$. Suppose that $y_{i^*} P_{i^*} x_{i^*}$ for some $i^* \in S_k(R)$. By definition of objective indifferences, $\eta(y_{i^*}) \neq \eta(x_{i^*})$. By Fact 1, $\{i \in N : x_i \in \eta(y_{i^*})\} \subseteq \cup_{\ell=1}^{k-1} S_\ell(R)$. By feasibility of y, if $\eta(y_{i^*}) \neq \eta(x_{i^*})$, there must be an agent j in $\cup_{\ell=1}^{k-1} S_\ell(R)$ for whom $x_j \in \eta(y_{i^*})$ but $y_j \notin \eta(y_{i^*})$. Therefore, $\neg(y_j I_j x_j)$, contradicting our assumption that $x_i I_i y_i$ for all $i \in \cup_{\ell=1}^{k-1} S_\ell(R)$. Thus we have that $x_i R_i y_i$ for all $i \in S_k(R)$. Now suppose there is some $i^* \in S_k(R)$ such that $x_{i^*} P_{i^*} y_{i^*}$. Then $S_k(R)$ and x block y, contradicting our assumption that y is in the core.

Thus $x_i I_i y_i$ for all $i \in N$, so x must also be in the core. (Since y was an arbitrary allocation in the core, this also proves Corollary 1.)

Next we show that for any domain $\tilde{\mathcal{R}}^n$ where $\tilde{\mathcal{R}}^n \nsubseteq \mathcal{R}(\mathcal{H})^n$ for any partition $\mathcal{H}$ of H, TTC$_{\succ}$ mechanisms are not CS on $\tilde{\mathcal{R}}^n$. If $\tilde{\mathcal{R}}^n \nsubseteq \mathcal{R}(\mathcal{H})^n$ for any $\mathcal{H}$, then $\tilde{\mathcal{R}}$ must contain two orderings, R_α and R_β, such that for some $h_1, h_2 \in H$ we have $h_1 I_\alpha h_2$ but $h_1 P_\beta h_2$. Without loss of generality, assume that $h_1 R_\beta h$ for all $h \in H$ such that $h I_\alpha h_2$. Also, without loss of generality, assume $w_i = h_i$ for all $i \in N$. Define $A = \{i \in N : w_i R_\alpha w_1\} \setminus \{2\}$ and consider the preference profile $R \in \tilde{\mathcal{R}}^n$ where $R_i = R_\alpha$ for all $i \in A$ and $R_i = R_\beta$ for all $i \in A^c$. Let $x = TTC_{\succ}(R)$. It follows directly from Lemma 1 that $x = w$. However, as we noted earlier, x is Pareto dominated by $y = (w_2, w_1, w_3, ..., w_n)$, and is therefore not in the core of the market. It remains to show that y is in the core.

Toward a contradiction, suppose there is a coalition Q and allocation z that blocks y. Let $W = \{i \in Q : z_i P_i y_i\}$, which must be non-empty.

Claim 1. $W \subseteq A^c$.

> *Proof.* Toward a contradiction, suppose $W_A := W \cap A$ is non-empty and take some $i^* \in W_A$ such that $w_{i^*} R_\alpha w_i$ for all $i \in W$. Since $z_{i^*} P_{i^*} y_{i^*}$ and $y_{i^*} R_{i^*} x_{i^*} R_{i^*} w_{i^*}$, $z_{i^*} P_{i^*} w_{i^*}$. Also, since $i^* \in A$, $R_{i^*} = R_\alpha$, so $z_{i^*} P_{i^*} w_{i^*}$ implies $z_{i^*} P_\alpha w_{i^*}$. Therefore, by feasibility of z and since $z_Q = w_Q$, if $z_{i^*} P_\alpha w_{i^*}$ there must be an agent $j \in Q$ for whom $w_j I_\alpha z_{i^*}$ but $\neg(w_j I_\alpha z_j)$. Note that $j \in A \setminus \{1\}$, since $i^* \in A$ and $w_j P_\alpha w_{i^*}$. Thus, $y_j = w_j$ and $R_j = R_\alpha$. Recall that $z_i R_i y_i$ for all $i \in Q$; therefore, $y_j = w_j$, $R_j = R_\alpha$, and $\neg(w_j I_\alpha z_j)$ imply that $z_j P_j y_j$. That is, $j \in W_A$. But this contradicts our assumption that $w_{i^*} R_\alpha w_i$ for all $i \in W$. □

Claim 2. $Q \cap A \neq \emptyset$.

> *Proof.* Toward a contradiction, suppose that $Q \subseteq A^c$. Therefore, $R_i = R_\beta$ for all $i \in Q$. Now, take some $i^* \in W$ such that $w_{i^*} R_\beta w_i$ for all $i \in W$. Since $i^* \in W$, we know that $z_{i^*} P_\beta y_{i^*}$. Also, since y

Pareto dominates x and x is individually rational, $y_{i^*} R_\beta x_{i^*} R_\beta w_{i^*}$. So $z_{i^*} P_\beta w_{i^*}$. By feasibility of z and since $z_Q = w_Q$, if $z_{i^*} P_\beta w_{i^*}$, there must exist an agent $j \in Q$ such that $w_j I_\beta z_{i^*}$ but $\neg(w_j I_\beta z_j)$. Since $j \in Q$, we have $z_j R_\beta y_j$. Therefore, since y Pareto dominates x and x is individually rational, $z_j R_\beta w_j$. So $\neg(w_j I_\beta z_j)$ implies $z_j P_\beta w_j$. If $j \in A^c \setminus \{2\}$, then $w_j = y_j$, in which case $j \in W$. However, this contradicts our assumption that $w_{i^*} R_\beta w_i$ for all $i \in W$. Therefore, it must be that $j = 2$. Since $Q \subseteq A^c$ and $z_Q = w_Q$, we know that $z_2 \neq w_1$. Also, since $2 \in Q$, we know that $z_2 R_\beta w_1 (= y_2)$. Consequently, by feasibility of z, there must be an agent $j^* \in Q$ such that $(y_{j^*} =) w_{j^*} R_\beta\ w_1$, but $\neg(z_{j^*} I_\beta w_{j^*})$. Since $j^* \in Q$, $z_{j^*} R_\beta w_{j^*}$, so $\neg(z_{j^*} I_\beta w_{j^*})$ implies $z_{j^*} P_\beta w_{j^*}$. But then $j^* \in W$ and $w_{j^*}\ I_\beta\ z_2\ R_\beta\ w_1\ P_\beta\ w_2\ P_\beta\ w_{i^*}$, again contradicting our assumption that $w_{i^*} R_\beta w_i$ for all $i \in W$. □

Without loss of generality, assume that the agents in Q form a single trading cycle under z.[5] Since Q contains agents in both A and A^c, there must be an agent $\bar{b} \in A^c$ such that $z_{\bar{b}} \in w_A$ and an agent $\bar{a} \in A$ such that $z_{\bar{a}} \in w_{A^c}$. In fact, $\bar{a}$ must be the only agent in A who receives a house from an agent in A^c. Recall that for every $i \in A^c \setminus \{2\}$, $w_1 P_\alpha w_i$. Therefore, if there exists some agent $a \neq \bar{a}$ in A such that $z_a \in w_{A^c}$, then either $w_1 P_\alpha z_{\bar{a}}$ or $w_1 P_\alpha z_a$. But since $a, \bar{a} \in A$, $w_a R_\alpha w_1$ and $w_{\bar{a}} R_\alpha w_1$, contradicting individual rationality of z for either a or $\bar{a}$. Also note that $z_{\bar{a}} = w_2$, since if $z_{\bar{a}} \in w_{A^c \setminus \{2\}}$ we would have $w_{\bar{a}} P_\alpha z_{\bar{a}}$, violating individual rationality of z. Moreover, $\bar{b}$ must be the only agent in A^c to receive a house from an agent in A, since only one agent in A gets a house from an agent in A^c and Q forms a single trading cycle. Therefore, we can write the trading cycle Q forms as

$$a_1 \to ... \to a_m \to \bar{a} \to 2 \to b_1 \to ... \to \bar{b} \to a_1$$

where $a_1, ..., a_m, \bar{a} \in A$ and $2, b_1, ..., \bar{b} \in A^c$.

Since $W \subseteq A^c$, we know that $z_a I_\alpha y_a$ for all $a \in \{a_1, ..., a_m, \bar{a}\}$. Also, recall that $y_i I_\alpha w_i$ for all $i \in A$. Therefore, $z_a I_\alpha w_a$ for all $a \in \{a_1, ..., a_m, \bar{a}\}$. In particular, since $z_{\bar{a}} = w_2$, we know that $w_{\bar{a}} I_\alpha w_2$. Also, since $z_{a_m} = w_{\bar{a}}$, $w_{a_m} I_\alpha w_{\bar{a}}$. Therefore, $w_{a_m} I_\alpha w_2$. By repeatedly applying the same reasoning, it is straightforward to show that $w_{a_1} I_\alpha w_2$. Also, we know that $z_b R_\beta y_b$ for all $b \in \{2, b_1, ..., \bar{b}\}$, with $z_b P_\beta y_b$ for at least one b since $W \neq \emptyset$. Recall that $y_2 = w_1$, so $(z_2 =) w_{b_1} R_\beta w_1$. Also, $(z_{b_1} =) w_{b_2} R_\beta w_{b_1}$, so $w_{b_2} R_\beta w_1$. By repeatedly applying the same reasoning, it is straightforward to show that $w_{a_1} P_\beta w_1$. But then $w_{a_1} P_\beta w_1$ and $w_{a_1} I_\alpha w_2$, contradicting our assumption that $1 R_\beta h$ for all $h I_\alpha w_2$. □

Proposition 1. *For any market, the weak core is non-empty and $TTC_\succ$ mechanisms select an allocation in the weak core.*

Proof. Fix a market (N, H, w, R) and a tie-breaking profile $\succ$. Let $x = TTC_\succ(R)$. Toward a contradiction, suppose there exists a coalition $Q \subseteq N$ and allocation y that weakly blocks x. That is, $y_Q = w_Q$ and $y_i P_i x_i$ for all $i \in Q$. Let k be the first step of $TTC_\succ(R)$ that an agent in Q is assigned; that is, $S_k(R) \cap Q \neq \emptyset$ and

[5] If the agents in Q formed two or more trading cycles, then some cycle C must contain an agent i such that $z_i P_i y_i$. Moreover, since $C \subseteq Q$, $z_j R_j y_j$ for all $j \in C$; therefore, it is without loss of generality to take $Q = C$.

$\cup_{\ell=1}^{k-1} S_\ell(R) \cap Q = \emptyset$. Take some $i \in S_k(R) \cap Q$. Since $y_Q = w_Q$, we know that $y_i = w_j$ for some $j \in Q$. But $y_i P_i x_i$ implies $j \in \cup_{\ell=1}^{k-1} S_\ell(R) \cap Q$, contradicting that $\cup_{\ell=1}^{k-1} S_\ell(R) \cap Q = \emptyset$. □

Appendix A.2 Group strategy-proofness

Theorem 2. *$\mathcal{R}^n$ is a symmetric-maximal domain on which $TTC_\succ$ mechanisms are group strategy-proof if and only if it is an objective indifferences domain.*

Before proving Theorem 2, we review an important property of $\text{TTC}_\succ$ and state a useful lemma. Let $L(h, R_i) = \{h' \in H : hR_ih'\}$ be the lower contour set of a preference ranking R_i at house h.

Monotonicity (MON). A rule f is **monotone** if $f(R) = f(R')$ for any preference profiles R and R' such that $L(f_i(R), R_i) \subseteq L(f_i(R), R'_i)$ for all $i \in N$.

That is, a rule f is monotone if, whenever any set of agents move their allocations upwards in their rankings, the allocation remains the same. It is straightforward to show that TTC is monotone for strict preferences; e.g. Takamiya (2001). Then, since $TTC_\succ(R) \equiv TTC(R_\succ)$ for any R and $\succ$, it follows directly that $\text{TTC}_\succ$ mechanisms are monotone.

The following result is adapted from Sandholtz and Tai (2024), who show it for TTC with strict preferences.

Lemma 2 (Sandholtz and Tai, 2024)**.** *For any R, R', let $x = TTC_\succ(R)$ and $y = TTC_\succ(R')$. Suppose there is some i such that $y_i P_{i,\succ} x_i$. Then there exists some agent j and house h such that $hP'_{j,\succ}x_j$ and $x_j P_{j,\succ} h$.*

We now provide our proof of Theorem 2.

Proof of Theorem 2. Fix (N, H, w). The result is trivial for $|N| = 1$, so assume $|N| \geq 2$. First we show that for any objective indifferences domain, $\text{TTC}_\succ$ mechanisms are GSP. Fix some tie-breaking rule $\succ$. Let $\mathcal{H}$ be any partition of H and let $R \in \mathcal{R}(\mathcal{H})^n$. If $\mathcal{H} = \{H\}$, the result is trivial, so suppose assume that $\mathcal{H}$ has at least two blocks. Suppose $Q \subseteq N$ reports R'_Q where $R' = (R'_Q, R_{-Q}) \in \mathcal{R}(\mathcal{H})^n$. Let $y = \text{TTC}_\succ(R')$. We will show that if $y_i P_i x_i$ for some $i \in Q$, then $x_j P_j y_j$ for some $j \in Q$. Let $R'' = (R''_Q, R_{-Q})$ be the preference profile in $\mathcal{R}(\mathcal{H})^n$ such that for each $i \in Q$, R''_i top-ranks the houses in $\eta(y_i)$ and otherwise preserves the ordering of R_i. That is, for any $h \in \eta(y_i)$ and $h' \notin \eta(y_i)$, $hP''_i h'$; otherwise, $hR''_i h'$ if and only if hR_ih'. Let $z = TTC_\succ(R'')$. By monotonicity of $\text{TTC}_\succ$, $z = y$. Take any $i^* \in Q$ such that $y_{i^*} P_{i^*} x_{i^*}$. Since $z = y$, this implies $z_{i^*} P_{i^*} x_{i^*}$, and consequently, $z_{i^*} P_{i^*,\succ_{i^*}} x_{i^*}$. Applying Lemma 2, there must be some $j \in Q$ and $h \in H$ such that $x_j P_{j,\succ_j} h$ but $hP''_{j,\succ_j} x_j$. Note that $h \notin \eta(x_j)$; if it were, then for any $R, R'' \in \mathcal{R}(\mathcal{H})^n$, $x_j P_{j,\succ_j} h$ if and only if $x_j P''_{j,\succ_j} h$. Therefore, $x_j P_j h$ and $hP''_j x_j$.[6] The only change from R_j to R''_j is to top-rank the houses in $\eta(y_j)$, so it must be that $h \in \eta(y_j)$. But then $x_j P_j y_j$, as desired.

Next we show that for any symmetric domain $\tilde{\mathcal{R}}^n$ where $\tilde{\mathcal{R}}^n \nsubseteq \mathcal{R}(\mathcal{H})^n$ for any $\mathcal{H}$, $\text{TTC}_\succ$ mechanisms are not GSP on $\tilde{\mathcal{R}}^n$. If $\tilde{\mathcal{R}}^n \nsubseteq \mathcal{R}(\mathcal{H})^n$ for any $\mathcal{H}$, then $\tilde{\mathcal{R}}$ must contain two orderings, R_α and R_β, such that for

[6]This is where the restriction to objective indifferences is used. Under general indifferences, this is not necessarily true.

some $h_1, h_2 \in H$ we have $h_1 I_\alpha h_2$ but $h_1 P_\beta h_2$. The symmetric requirement also necessitates that $\tilde{\mathcal{R}}$ contains some R_γ such that $h_2 P_\gamma h_1$. Taking only the existence of $R_\alpha, R_\beta, R_\gamma \in \tilde{\mathcal{R}}$ for granted, we find a preference profile $R \in \tilde{\mathcal{R}}^n$ and tie-breaking profile $\succ$ such that $TTC_\succ(R)$ is not GSP.

Without loss of generality, assume $w_i = h_i$ for all $i \in N$. Define $A = \{i \in N : w_i R_\alpha w_1\} \setminus \{2\}$, $B = \{i \in A^c : w_i R_\beta w_1\} \cup \{2\}$, and $C = N \setminus (A \cup B)$. Consider the preference profile $R \in \tilde{\mathcal{R}}^n$ where

$$R_i = \begin{cases} R_\alpha & \text{if } i \in A \\ R_\beta & \text{if } i \in B \\ R_\gamma & \text{if } i \in C. \end{cases}$$

Note that $1 \in A$ and $2 \in B$, so $R_1 = R_\alpha$ and $R_2 = R_\beta$. Let $\succ$ be any tie-breaking profile such that for all $i \in N$, $i \succ_i j$ for all $j \neq i$. Also, let $2 \succ_1 j$ for all $j \neq 1, 2$. Let $x = TTC_\succ(R)$.

Claim 3. *$x_1 = w_1$ and $w_1 P_2 x_2$.*

Proof of Claim 3. It follows directly from Lemma 1 that $x_i = w_i$ for all $i \in (A \cup B) \setminus \{2\}$. In particular, $x_1 = w_1$. Consider the agent $j \in N$ such that $x_2 = w_j$. Since $x_i = w_i$ for all $i \in (A \cup B) \setminus \{2\}$, we know that $j \in C \cup \{2\}$. Therefore, $w_1 P_\beta w_j$, so $w_1 P_2 x_2$. □

Now suppose that agent 1 misreports $R'_1 = R_\gamma$. Let $R' = (R'_1, R_{-1})$ and let $y = TTC_\succ(R')$.

Claim 4. *$y_1 = w_2$ and $y_2 = w_1$.*

Proof of Claim 4. It follows directly from Lemma 1 that $y_i = w_i$ for all $i \in (A \cup B) \setminus \{1, 2\}$. Moreover, note that $y_i = w_i$ for all $i \in C$ such that $w_i P_\gamma w_2$. To see this, suppose $W = \{i \in C : w_i P_\gamma w_2, y_i \neq w_i\}$ is non-empty. Take any $i^* \in W$ such that $w_{i^*} R_\gamma w_i$ for all $i \in W$. By individual rationality of y, $y_{i^*} R_{i^*} w_{i^*}$. Also, since $i^* \succ_{i^*} i$ for all $i \neq i^*$, by definition of $\text{TTC}_\succ$ we know that if $y_{i^*} I_{i^*} w_{i^*}$ then $y_{i^*} = w_{i^*}$. Therefore, $y_{i^*} R_{i^*} w_{i^*}$ implies $y_{i^*} P_{i^*} w_{i^*}$. Since $i^* \in C$, $R_{i^*} = R_\gamma$, so $y_{i^*} P_\gamma w_{i^*}$. Consider the agent j such that $y_{i^*} = w_j$. We know that $j \notin (A \cup B) \setminus \{1, 2\}$, because $y_j \neq w_j$ and we showed that $y_i = w_i$ for all $i \in (A \cup B) \setminus \{1, 2\}$. Moreover, $j \notin \{1, 2\}$ because $w_j P_\gamma w_{i^*} P_\gamma w_2 P_\gamma w_1$. Therefore, $j \in C$. Also, $j \in W$. But this contradicts our assumption that $w_{i^*} R_\gamma w_i$ for all $i \in W$.

Toward a contradiction, suppose that $y_1 \neq w_2$. Since $y_i = w_i$ for all $i \in (A \cup B) \setminus \{1, 2\}$ and all $i \in C$ such that $w_i P_\gamma w_2$, it must be that $w_2 R_\gamma y_1$. Recall that $2 \succ_1 i$ for all $i \neq 1, 2$. Therefore, if $w_2 R_\gamma y_1$, then $2 \in S_k(R')$ and $1 \in S_\ell(R')$ for $k < \ell$. That is, agent 2 must have been assigned at an earlier step of $TTC_\succ(R')$ than agent 1 was; otherwise, agent 1 would have pointed at agent 2 at step ℓ when 1 was assigned to y_1. This implies that $y_2 R_2 w_1$ and $y_2 \neq w_1$. But $R_2 = R_\beta$, so $y_2 R_\beta w_1$. Then $y_2 = w_j$ for some $j \in (A \cup B) \setminus \{1, 2\}$, contradicting $y_i = w_i$ for all i in $(A \cup B) \setminus \{1, 2\}$. □

So $(w_2 =) y_1 I_1 x_1 (= w_1)$ and $(w_1 =) y_2 P_2 x_2$ for $Q = \{1, 2\}$, meaning $TTC_\succ(R)$ is not GSP. □

Appendix B Relation to school choice with priorities

We briefly note that TTC in the objective indifferences setting is not identical to TTC in the school choice with priorities setting. Intuitively, in Shapley-Scarf markets with objective indifferences, the fixed tie-breaking rule determines who the agents point at; conversely, a school priority ranking determines which schools point at i. Consider an example with 3 schools and 4 students.

Example 7. Let the set of schools (objects) be $H = \{A, B, C\}$, with C having two slots. Let the students be $N = \{a, b, c_1, c_2\}$, where a is "endowed" with A, and so on. Below are a possible school priority ranking for the school choice setting and a possible tie-breaking profile for the Shapley-Scarf setting.

A	B	C
a	b	c_1
b	a	c_2
c_1	c_2	a
c_1	c_1	b

School priority for school choice setting

vs

$\succ_a$	$\succ_b$	$\succ_{c_1}$	$\succ_{c_{c_2}}$
c_1	c_2	c_1	c_1
c_2	c_1	c_2	c_2
a	a	a	a
b	b	b	b

Tie-breaking profile for Shapley-Scarf setting

Consider the preference profiles $R = (R_a, R_b, R_{c_1}, R_{c_2})$ and $R' = (R'_a, R'_b, R'_{c_1}, R'_{c_2})$, shown below.

R_a	R_b	R_{c_1}	R_{c_2}
C	C	A	A
A	A	B	B
B	B	C	C

Preference profile R'

R'_a	R'_b	R'_{c_1}	R'_{c_2}
C	C	B	B
A	A	A	A
B	B	C	C

Preference profile R'

Note that under R and R', both c_1 and c_2 have the same preferences. TTC with school priorities results in $(A : c_1, B : c_2, C : ab)$ and $(A : c_2, B : c_1, C : ab)$ under R and R' respectively. Note that c_1 gets his preferred school in either case, since his priority at school C is higher than c_2's priority at school C. In fact, in the school choice setting, since c_1 has higher priority at C than c_2 has, whenever c_1 and c_2 have the same preferences c_1 will weakly prefer his assignment to c_2's assignment. By contrast, in the Shapley-Scarf setting, $\text{TTC}_{\succ}$ results in $(A : c_1, B : c_2, C : ab)$ and $(A : c_1, B : c_2, C : ab)$ under R and R' respectively. Now, c_1 does not necessarily always get his top choice when c_1 and c_2 have the same preferences. Under R', c_2 gets his top choice at the expense of c_1, since $c_2 \succ_b c_1$.